Suggested Solutions to:
Resit Exam, Fall 2018
Contract Theory
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Question 1: Optimal taxation

Part (a)

Derive the first-best solution; that is, characterize the optimal menu of contracts under the assumption that the government actually can observe whether any given citizen is able or not. Explain the economic intuition behind your result.

- We can define the first-best solution as the principal's optimal choice if being able to observe θ. Without uncertainty, the IC constraints are irrelevant. Moreover, by assumption, there are no participation constraints (see the question). Hence, the principal's problem can be written as (here I do not plug in the functional form $C(q,\theta) = \theta q$ until the end)

$$\max_{\underline{t},\underline{q},\bar{t},\bar{q}} \left\{ \nu G \left[\underline{t} - C\left(\underline{q}, \underline{\theta}\right) \right] + (1-\nu) G \left[\bar{t} - C\left(\bar{q}, \bar{\theta}\right) \right] \right\}$$

subject to

$$\nu S\left(\underline{q}\right) + (1-\nu) S(\bar{q}) \geq \nu \underline{t} + (1-\nu) \bar{t}. \qquad \text{(Budget)}$$

The Lagrangian is:

$$\begin{aligned} \mathcal{L} &= \nu G \left[\underline{t} - C\left(\underline{q}, \underline{\theta}\right) \right] + (1-\nu) G \left[\bar{t} - C\left(\bar{q}, \bar{\theta}\right) \right] \\ &\quad + \mu \left[\nu S\left(\underline{q}\right) + (1-\nu) S(\bar{q}) - \nu \underline{t} - (1-\nu) \bar{t} \right]. \end{aligned}$$

FOC w.r.t. $\underline{t}$:

$$\frac{\partial \mathcal{L}}{\partial \underline{t}} = 0 \Leftrightarrow G' \left[\underline{t} - C\left(\underline{q}, \underline{\theta}\right) \right] = \mu,$$

which implies, as $G' > 0$, that the budget constraint binds: $\mu > 0$. FOC w.r.t. $\underline{q}$:

$$\frac{\partial \mathcal{L}}{\partial \underline{q}} = 0 \Leftrightarrow G' \left[\underline{t} - C\left(\underline{q}, \underline{\theta}\right) \right] C_q\left(\underline{q}, \underline{\theta}\right) = \mu S'\left(\underline{q}\right).$$

Combining the two FOCs yields:

$$C_q\left(\underline{q}^{FB}, \underline{\theta}\right) = S'\left(\underline{q}^{FB}\right)$$

or

$$S'\left(\underline{q}^{FB}\right) = \underline{\theta}$$

if we use the functional form $C(q, \theta) = \theta q$. This defines the good type's first-best quantity. FOC w.r.t. $\bar{t}$:

$$\frac{\partial \mathcal{L}}{\partial \bar{t}} = 0 \Leftrightarrow G'\left[\bar{t} - C\left(\bar{q}, \bar{\theta}\right)\right] = \mu.$$

FOC w.r.t. $\bar{q}$:

$$\frac{\partial \mathcal{L}}{\partial \bar{q}} = 0 \Leftrightarrow G'\left[\bar{t} - C\left(\bar{q}, \bar{\theta}\right)\right] C_q\left(\bar{q}, \bar{\theta}\right) = \mu S'\left(\bar{q}\right).$$

Combining the two FOCs yields:

$$C_q\left(\bar{q}^{FB}, \bar{\theta}\right) = S'\left(\bar{q}^{FB}\right)$$

or, if we use the functional form $C(q, \theta) = \theta q$,

$$S'\left(\bar{q}^{FB}\right) = \bar{\theta}.$$

This defines the bad type's first-best quantity. Combining the FOCs w.r.t. $\underline{t}$ and $\bar{t}$, we also have

$$G'\left[\underline{t} - C\left(\underline{q}, \underline{\theta}\right)\right] = \mu = G'\left[\bar{t} - C\left(\bar{q}, \bar{\theta}\right)\right]$$

By $G'' < 0$, this implies

$$\underline{t}^{FB} - C\left(\underline{q}^{FB}, \underline{\theta}\right) = \bar{t}^{FB} - C\left(\bar{q}^{FB}, \bar{\theta}\right). \tag{1}$$

We can conclude that: (i) At the first-best solution, the two types of citizens get *the same level of utility*. This is because the government has a concave utility function (G): it dislikes inequality. (ii) There is *no trade-off* between efficiency and equity—the government can achieve both objectives perfectly.

- As the budget constraint is binding, we have:

$$\nu S\left(\underline{q}^{FB}\right) + (1-\nu) S\left(\bar{q}^{FB}\right) = \nu \underline{t}^{FB} + (1-\nu)\bar{t}^{FB}. \tag{2}$$

The equations (1) and (2) can be solved for $\underline{t}^{FB}$ and $\bar{t}^{FB}$:

$$\underline{t}^{FB} = C\left(\underline{q}^{FB}, \underline{\theta}\right) + A \qquad \text{and} \qquad \bar{t}^{FB} = C\left(\bar{q}^{FB}, \bar{\theta}\right) + A,$$

where A is the is aggregate surplus:

$$\begin{aligned} A \equiv\ & \nu\left[S\left(\underline{q}^{FB}\right) - C\left(\underline{q}^{FB}, \underline{\theta}\right)\right] \\ & + (1-\nu)\left[S\left(\bar{q}^{FB}\right) - C\left(\bar{q}^{FB}, \bar{\theta}\right)\right]. \end{aligned}$$

  - Each type of citizen gets compensated for their costs of providing work effort and in addition receives an amount corresponding to the aggregate surplus.

Part (b)

Show that there is "efficiency at the top" (i.e., that $\underline{q}^{SB} = \underline{q}^{FB}$). Also show that the second-best quantity of the agent-type that is "not able" is lower than his first-best quantity (i.e., that $\overline{q}^{SB} < \overline{q}^{FB}$). Explain the nature of the trade-off that the government faces.

- Under second best the government must take the IC constraints into account (again I wait until the end with plugging in the functional form $C(q, \theta) = \theta q$):

$$\underline{t} - C\left(\underline{q}, \underline{\theta}\right) \geq \overline{t} - C\left(\overline{q}, \underline{\theta}\right), \tag{IC-good}$$

$$\overline{t} - C\left(\overline{q}, \overline{\theta}\right) \geq \underline{t} - C\left(\underline{q}, \overline{\theta}\right). \tag{IC-bad}$$

  Therefore, the government's problem is to choose $\underline{t}, \underline{q}, \overline{t}, \overline{q}$ so as to maximize

$$V = \nu G\left[\underline{t} - C\left(\underline{q}, \underline{\theta}\right)\right] + (1-\nu) G\left[\overline{t} - C\left(\overline{q}, \overline{\theta}\right)\right],$$

  subject to the two IC constraints and the budget constraint. One can solve the problem by guessing that the bad type's IC constraint does not bind and then check this afterwards—however, as the question is stated we are allowed to just take for granted that IC-bad does not bind (not having to check afterwards).

- The problem can now be written as

$$\max_{\underline{t}, \underline{q}, \overline{t}, \overline{q}} \left\{\nu G\left[\underline{t} - C\left(\underline{q}, \underline{\theta}\right)\right] + (1-\nu) G\left[\overline{t} - C\left(\overline{q}, \overline{\theta}\right)\right]\right\}$$

  subject to

$$\nu S\left(\underline{q}\right) + (1-\nu) S\left(\overline{q}\right) \geq \nu \underline{t} + (1-\nu)\overline{t}, \tag{Budget}$$

$$\underline{t} - C\left(\underline{q}, \underline{\theta}\right) \geq \overline{t} - C\left(\overline{q}, \underline{\theta}\right). \tag{IC-good}$$

- The Lagrangian is:

$$\begin{aligned}
\mathcal{L} \quad = \quad & \nu G\left[\underline{t} - C\left(\underline{q}, \underline{\theta}\right)\right] + (1-\nu) G\left[\overline{t} - C\left(\overline{q}, \overline{\theta}\right)\right] \\
& + \mu\left[\nu S\left(\underline{q}\right) + (1-\nu) S\left(\overline{q}\right) - \nu \underline{t} - (1-\nu)\overline{t}\right] \\
& + \lambda\left[\underline{t} - C\left(\underline{q}, \underline{\theta}\right) - \overline{t} + C\left(\overline{q}, \underline{\theta}\right)\right].
\end{aligned}$$

- FOC w.r.t. $\underline{t}$:

$$\frac{\partial \mathcal{L}}{\partial \underline{t}} = 0 \Leftrightarrow \nu G'\left[\underline{t} - C\left(\underline{q}, \underline{\theta}\right)\right] = \nu\mu - \lambda. \tag{3}$$

  Hence, as $G' > 0$ and $\lambda \geq 0$, the budget constraint binds: $\mu > 0$.

- FOC w.r.t. $\underline{q}$:

$$\frac{\partial \mathcal{L}}{\partial \underline{q}} = 0 \Leftrightarrow \nu G'\left[\underline{t} - C\left(\underline{q}, \underline{\theta}\right)\right] C_q\left(\underline{q}, \underline{\theta}\right) = \nu\mu S'\left(\underline{q}\right) - \lambda C_q\left(\underline{q}, \underline{\theta}\right).$$

- Combining these two FOCs yields:

$$C_q\left(\underline{q}^{SB},\underline{\theta}\right) = S'\left(\underline{q}^{SB}\right)$$

or

$$S'\left(\underline{q}^{SB}\right) = \underline{\theta}.$$

Conclusion: $\underline{q}^{SB} = \underline{q}^{FB}$, that is, "efficiency at the top".

- FOC w.r.t. $\bar{t}$:

$$\frac{\partial \mathcal{L}}{\partial \bar{t}} = 0 \Leftrightarrow (1-\nu)\, G'\left[\bar{t} - C\left(\bar{q},\bar{\theta}\right)\right] = (1-\nu)\,\mu + \lambda.$$

  - Recall that the FOC w.r.t. $\underline{t}$ gave us $\nu G'\left[\underline{t} - C\left(\underline{q},\underline{\theta}\right)\right] = \nu\mu - \lambda$.
  - Add these two FOCs:

$$\mu = \nu G'\left[\underline{t} - C\left(\underline{q},\underline{\theta}\right)\right] + (1-\nu)\, G'\left[\bar{t} - C\left(\bar{q},\bar{\theta}\right)\right]. \tag{4}$$

- FOC w.r.t. $\bar{q}$: $\frac{\partial \mathcal{L}}{\partial \bar{q}} = 0 \Leftrightarrow$

$$(1-\nu)\, G'\left(\overline{U}\right) C_q\left(\bar{q},\bar{\theta}\right) = \mu\,(1-\nu)\, S'\left(\bar{q}\right) + \lambda C_q\left(\bar{q},\underline{\theta}\right).$$

Rewrite this:

$$\begin{aligned}
\mu\,(1-\nu)\, S'\left(\bar{q}\right) &= \left[\overbrace{(1-\nu)\, G'\left(\overline{U}\right) + \nu G'\left(\underline{U}\right)}^{=\mu \text{ by } (4)} - \overbrace{\nu G'\left(\underline{U}\right)}^{=\nu\mu-\lambda \text{ by } (3)}\right] C_q\left(\bar{q},\bar{\theta}\right) - \lambda C_q\left(\bar{q},\underline{\theta}\right) \\
&= \left[\mu\,(1-\nu) + \lambda\right] C_q\left(\bar{q},\bar{\theta}\right) - \lambda C_q\left(\bar{q},\underline{\theta}\right)
\end{aligned}$$

or

$$\begin{aligned}
S'\left(\bar{q}^{SB}\right) &= C_q\left(\bar{q}^{SB},\bar{\theta}\right) + \frac{\lambda^{SB}\left[C_q\left(\bar{q}^{SB},\bar{\theta}\right) - C_q\left(\bar{q}^{SB},\underline{\theta}\right)\right]}{(1-\nu)\,\mu^{SB}} \\
&= \bar{\theta} + \frac{\lambda^{SB}\left(\bar{\theta} - \underline{\theta}\right)}{(1-\nu)\,\mu^{SB}}
\end{aligned}$$

Hence, $\bar{q}^{SB} < \bar{q}^{FB}$ if IC-good binds ($\lambda^{SB} > 0$). So does it?

- **Claim**: $\lambda > 0$ at the optimum. **Proof**. From (4) we have that

$$\mu = \nu G'\left(\underline{U}\right) + (1-\nu)\, G'\left(\overline{U}\right),$$

Plugging μ into the FOC w.r.t. $\underline{t}$, $\lambda = \nu\mu - \nu G'\left(\underline{U}\right)$, we have:

$$\lambda = \nu\,(1-\nu)\left[G'\left(\overline{U}\right) - G'\left(\underline{U}\right)\right].$$

To prove that $\lambda > 0$, suppose not so that $\lambda = 0$ and let us derive a contradiction. From the above expression for λ we then have

$$G'\left(\overline{U}\right) = G'\left(\underline{U}\right) \qquad \Rightarrow \qquad \bar{t} - C\left(\bar{q},\bar{\theta}\right) = \underline{t} - C\left(\underline{q},\underline{\theta}\right).$$

If the IC-good constraint does not bind (i.e., if $\lambda = 0$), then

$$\underline{t} - C\left(\underline{q}, \underline{\theta}\right) > \bar{t} - C\left(\overline{q}, \underline{\theta}\right).$$

But this together with the above equality implies $\bar{t} - C\left(\overline{q}, \overline{\theta}\right) > \bar{t} - C\left(\overline{q}, \underline{\theta}\right)$ or $C\left(\overline{q}, \underline{\theta}\right) > C\left(\overline{q}, \overline{\theta}\right)$—an impossibility. □

- Restate the (rewritten) FOC w.r.t. $\overline{q}$:

$$S'\left(\overline{q}^{SB}\right) = C_q\left(\overline{q}^{SB}, \overline{\theta}\right) + \frac{\lambda^{SB}\left[C_q\left(\overline{q}^{SB}, \overline{\theta}\right) - C_q\left(\overline{q}^{SB}, \underline{\theta}\right)\right]}{(1-\nu)\,\mu^{SB}}.$$

  - Given that $\lambda^{SB} > 0$, we have $\overline{q}^{SB} < \overline{q}^{FB}$. Previous analysis also showed that $\underline{U} > \overline{U}$.
  - Implication: there is a trade-off between efficiency and equality.

- Interpretation:

  - The bad type works too little: $\overline{q}^{SB} < \overline{q}^{FB}$.

Question 2: Moral hazard with three effort and output levels

Part (a)

Consider the case where P induces A to choose the effort $e = 1$. Derive, for this case, the optimal payment levels (any method—graphical or non-graphical—is fine, as long as the results are shown). Moreover, compute P's maximized second-best payoff (i.e, the expected output minus the expected payments) if inducing $e = 1$, and denote this by V_1^{SB}. What is the condition required for having $V_1^{SB} \geq V_0^{SB}$? Interpret this condition.

If P wants to induce A to choose $e = 1$, then P's profit-maximization problem can be written as:

$$\max_{t_S, t_M, t_L} (\pi_S - a)(q_S - t_S) + (\pi_M + a)(q_M - t_M) + \pi_L (q_L - t_L)$$

subject to

$$(\pi_S - a) t_S + (\pi_M + a) t_M + \pi_L t_L - \Psi \geq 0, \tag{IR}$$

$$(\pi_S - a) t_S + (\pi_M + a) t_M + \pi_L t_L - \Psi \geq \pi_S t_S + \pi_M t_M + \pi_L t_L \Leftrightarrow t_M \geq t_S + \frac{\Psi}{a}, \tag{IC-0}$$

$$\begin{aligned} & (\pi_S - a) t_S + (\pi_M + a) t_M + \pi_L t_L - \Psi \\ \geq \quad & (\pi_S - a - b) t_S + (\pi_M + a) t_M + (\pi_L + b) t_L - 2\Psi \\ \Leftrightarrow \quad & \frac{\Psi}{b} + t_S \geq t_L, \end{aligned} \tag{IC-2}$$

$$t_S \geq 0, \qquad t_M \geq 0, \qquad t_L \geq 0. \tag{LL}$$

We can find the solution to this problem by reasoning as follows:

- By inspection, the IR constraint is implied by IC-0 and the three LL constraints. We can therefore ignore IR.
- At the optimum, we must have $t_L = 0$.
  - *Proof.* Suppose, per contra, that $t_L > 0$ at the optimum. Then we could lower t_L while still not violating neither LL nor IC-2 (and t_L does not appear at all in IC-0). This would increase the value of the objective, which contradicts the assumption that we started out at an optimum. □
- Plugging in $t_L = 0$ in the problem above yields the following new problem:

$$\max_{t_S, t_M} (\pi_S - a)(q_S - t_S) + (\pi_M + a)(q_M - t_M) + \pi_L q_L$$

subject to

$$t_M \geq t_S + \frac{\Psi}{a}, \tag{IC-0}$$

$$\frac{\Psi}{b} + t_S \geq 0, \tag{IC-2}$$

$$t_S \geq 0, \qquad t_M \geq 0. \tag{LL}$$

- By inspection, the constraint IC-2 is implied by the LL constraint $t_S \geq 0$. We can therefore ignore IC-2.

- At the optimum (of the new problem, and therefore also of the original one), we must have $t_S = 0$.

  - *Proof.* Suppose, per contra, that $t_S > 0$ at the optimum. Then we could lower t_S while still not violating neither the LL constraint $t_S \geq 0$ nor the IC-0 constraint. This would increase the value of the objective, which contradicts the assumption that we started out at an optimum. □

- Next, the result that $t_S = 0$ enables us to simplify the problem further:

$$\max_{t_M} (\pi_S - a) q_S + (\pi_M + a)(q_M - t_M) + \pi_L q_L$$

  subject to

$$t_M \geq \frac{\Psi}{a}, \tag{IC-0}$$

$$t_M \geq 0. \tag{LL}$$

  Note that LL is implied by IC-0. Indeed, it is clear from inspection that the optimal value of t_M is such that IC-0 binds: $t_M = \Psi/a$.

- Summing up, we can conclude that the solution to the maximization problem is $t_S = t_L = 0$ and $t_M = \Psi/a$.

- We can now also write up an expression for P's maximized second-best payoff, when inducing $e = 1$:

$$\begin{aligned} V_1^{SB} &= (\pi_S - a)(q_S - t_S) + (\pi_M + a)(q_M - t_M) + \pi_L (q_L - t_L) \\ &= (\pi_S - a) q_S + (\pi_M + a)\left(q_M - \frac{\Psi}{a}\right) + \pi_L q_L. \end{aligned}$$

- This in turn means that the condition that is required for $V_1^{SB} \geq V_0^{SB}$, is given by

$$\begin{aligned} V_1^{SB} &\geq V_0^{SB} \Leftrightarrow \\ (\pi_S - a) q_S + (\pi_M + a)\left(q_M - \frac{\Psi}{a}\right) + \pi_L q_L &\geq \pi_S q_S + \pi_M q_M + \pi_L q_L \Leftrightarrow \\ a(q_M - q_S) &\geq \pi_M \frac{\Psi}{a} + \Psi. \end{aligned}$$

  - Interpretation: The benefit with inducing $e = 1$ instead of $e = 0$ is that then, with probability a, the machine produces the quantity q_M instead of q_S. This benefit is captured by the left-hand side of the last inequality above. There are two kinds of costs of inducing $e = 1$ instead of $e = 0$: A must be compensated for the direct cost of making a small effort rather than no effort (Ψ); in addition, there is an informational cost ($\pi_M \frac{\Psi}{a}$) that is due to the moral hazard feature of the problem (P must ensure that the IC constraints hold). These two kinds of cost are captured by the right-hand side of the last inequality.

Part (b)

Consider the case where P induces A to choose the effort $e = 2$. Derive, for this case, the optimal payment levels (any method—graphical or non-graphical—is fine, as long as the results are shown). Moreover, compute P's maximized second-best payoff (i.e, the expected output minus the expected payments) if inducing $e = 2$, and denote this by V_2^{SB}. What is the condition required for having $V_2^{SB} \geq V_0^{SB}$ and $V_2^{SB} \geq V_1^{SB}$? Interpret these conditions.

If P wants to induce A to choose $e = 2$, then P's profit-maximization problem can be written as follows:

$$\max_{t_S, t_M, t_L} (\pi_S - a - b)(q_S - t_S) + (\pi_M + a)(q_M - t_M) + (\pi_L + b)(q_L - t_L)$$

subject to

$$(\pi_S - a - b) t_S + (\pi_M + a) t_M + (\pi_L + b) t_L - 2\Psi \geq 0, \tag{IR}$$

$$\begin{aligned} & (\pi_S - a - b) t_S + (\pi_M + a) t_M + (\pi_L + b) t_L - 2\Psi \\ \geq\ & \pi_S t_S + \pi_M t_M + \pi_L t_L \\ \Leftrightarrow\ & a t_M + b t_L \geq (a + b) t_S + 2\Psi, \end{aligned} \tag{IC-0}$$

$$\begin{aligned} & (\pi_S - a - b) t_S + (\pi_M + a) t_M + (\pi_L + b) t_L - 2\Psi \\ \geq\ & (\pi_S - a) t_S + (\pi_M + a) t_M + \pi_L t_L - \Psi \\ \Leftrightarrow\ & b t_L \geq \Psi + b t_S, \end{aligned} \tag{IC-1}$$

$$t_S \geq 0, \qquad t_M \geq 0, \qquad t_L \geq 0. \tag{LL}$$

We can find the solution to this problem by reasoning as follows:

- By inspection, the constraint IR is implied by IC-0 and the three LL constraints. We can therefore ignore IR.
- At the optimum, we must have $t_S = 0$.
  - *Proof.* Suppose, per contra, that $t_S > 0$ at the optimum. Then we could lower t_S while still not violating neither the LL constraint nor the IC-1 and IC-0 constraints. This would increase the value of the objective, which contradicts the assumption that we started out at an optimum. □
- Plugging in $t_S = 0$ in the problem above yields the following new problem:

$$\max_{t_M, t_L} (\pi_S - a - b) q_S + (\pi_M + a)(q_M - t_M) + (\pi_L + b)(q_L - t_L)$$

subject to

$$a t_M + b t_L \geq 2\Psi, \tag{IC-0}$$

$$b t_L \geq \Psi, \tag{IC-1}$$

$$t_M \geq 0, \qquad t_L \geq 0. \tag{LL}$$

- By inspection, the LL constraint $t_L \geq 0$ is implied IC-1. We can therefore ignore the constraint $t_L \geq 0$.

- The remaining problem can most easily be solved with a graphical argument. The IC-0 and IC-1 constraints can be written as

$$t_L \geq \frac{2\Psi}{b} - \frac{a}{b} t_M \qquad \text{and} \qquad t_L \geq \frac{\Psi}{b},$$

  respectively. These inequalities as well as $t_L \geq 0$ are graphed in the attached figure. The constraints are satisfied in the yellow area. The equation of an iso-cost curve is given by

$$t_L = \frac{C}{\pi_L + b} - \frac{\pi_M + a}{\pi_L + b} t_M.$$

  P is better off with an outcome that is as close to the origin as possible. Therefore, if the slope of the iso-cost curve is steeper than IC-0 is, then it follows from the figure that the optimum is at $(t_M, t_L) = \left(0, \frac{2\Psi}{b}\right)$. Similarly, if the slope of the iso-cost curve is flatter than IC-0 is, then the optimum is at $(t_M, t_L) = \left(\frac{\Psi}{a}, \frac{\Psi}{b}\right)$. The condition for the iso-cost curve to be flatter than IC-0 is that:

$$\frac{\pi_M + a}{\pi_L + b} < \frac{a}{b} \Leftrightarrow b\pi_M < a\pi_L,$$

  which indeed holds by equation (1) in the exam question. We can thus conclude that optimum is at $(t_M, t_L) = \left(\frac{\Psi}{a}, \frac{\Psi}{b}\right)$.

- We can now also write up an expression for P's maximized second-best payoff, when inducing $e = 2$:

$$\begin{aligned} V_2^{SB} &= (\pi_S - a - b)(q_S - t_S) + (\pi_M + a)(q_M - t_M) + (\pi_L + b)(q_L - t_L) \\ &= (\pi_S - a - b)\, q_S + (\pi_M + a)\left(q_M - \frac{\Psi}{a}\right) + (\pi_L + b)\left(q_L - \frac{\Psi}{b}\right). \end{aligned}$$

- This in turn means that the condition that is required for $V_2^{SB} \geq V_0^{SB}$, is given by

$$\begin{aligned} V_2^{SB} &\geq V_0^{SB} \Leftrightarrow \\ (\pi_S - a - b)\, q_S + (\pi_M + a)\left(q_M - \frac{\Psi}{a}\right) + (\pi_L + b)\left(q_L - \frac{\Psi}{b}\right) &\geq \pi_S q_S + \pi_M q_M + \pi_L q_L \Leftrightarrow \\ a q_M + b q_L - (a + b)\, q_S &\geq (\pi_M + a)\frac{\Psi}{a} + (\pi_L + b)\frac{\Psi}{b} \Leftrightarrow \\ a q_M + b q_L - (a + b)\, q_S &\geq \pi_M \frac{\Psi}{a} + \pi_L \frac{\Psi}{b} - 2\Psi \end{aligned}$$

  - Interpretation: The benefit with inducing $e = 2$ instead of $e = 0$ is that then, with probability a, the machine produces the quantity q_M instead of q_S; and with probability b, it produces the quantity q_L instead of q_S. This benefit is captured by the left-hand side of the last inequality. There are two kinds of cost of inducing $e = 2$ instead of $e = 0$: A must be compensated for the direct cost of making a large effort rather than no effort (2Ψ); in addition, there is an informational cost ($\pi_M \frac{\Psi}{a} + \pi_L \frac{\Psi}{b}$) that is due to the moral hazard feature of the problem (P must ensure that the IC constraints hold). These two cost terms are captured by the right-hand side of the last inequality.

- Finally, the condition that is required for $V_2^{SB} \geq V_1^{SB}$, is given by

$$V_2^{SB} \geq V_1^{SB} \Leftrightarrow (\pi_S - a - b)\, q_S + (\pi_M + a) \left(q_M - \frac{\Psi}{a} \right) + (\pi_L + b) \left(q_L - \frac{\Psi}{b} \right)$$
$$\geq (\pi_S - a)\, q_S + (\pi_M + a) \left(q_M - \frac{\Psi}{a} \right) + \pi_L q_L \Leftrightarrow$$
$$-bq_S + bq_L - (\pi_L + b)\, \frac{\Psi}{b} \geq 0 \Leftrightarrow b\,(q_L - q_S) \geq \pi_L \frac{\Psi}{b} + \Psi$$

  - Interpretation: The benefit with inducing $e = 2$ instead of $e = 1$ is that then, with probability b, the machine produces the quantity q_L instead of q_S. This benefit is captured by the left-hand side of the last inequality. There are two kinds of cost of inducing $e = 2$ instead of $e = 1$: A must be compensated for the direct cost of making a large effort rather than a small effort (Ψ); in addition, there is an informational cost ($\pi_L \frac{\Psi}{b}$) that is due to the moral hazard feature of the problem (P must ensure that the IC constraints hold). These costs are captured by the right-hand side of the last inequality.